MATHEMATICS DEPARTMENT

"A computer is the mathematicians best friend"

μ - Games
Mathematics Utrecht

November 2023

Rules:

The idea of this event is to gap the bridge between mathematics and programming. When working on these exercises, we hope the participant will get a better understanding of the underlying mathematical concepts. You will not be required to do a lot of difficult programming. With array manipulation and basic functionality, you should be able to solve all the exercises.

When working on these exercises, you must conform to the following rules.

- You are allowed to work in groups of maximum 4 persons.
- You will have 3 hours to solve the problems.
- For the problems, you can use the default mathematics library of your programming language (for example *import math* in Python).
- You cannot look up any computer code that may help you with solving the problem.

After 3 hours, the solutions to the exercises will be discussed. To check your own solution, one can go to the website `http://clover.science.uu.nl/dj`.

The function *pow(n,p,m)* in Python may prove useful; it provides an efficient way to compute $n^p \pmod{m}$.

Problem 1: Palindromial Polynomials

Difficulty: ★ ☆ ☆ ☆ ☆
Key words: Polynomials

Given a palindromial polynomial in $Z[X]$ of degree $n \geq 2$. That is

$$P(X) = a_n X^n + a_{n-1} X^{n-1} + ... + a_1 X + a_0$$

such that $a_k = a_{n-k}$ for $0 \leq k \leq n$. Then, given a non-zero root $\alpha \neq \pm 1$ of P, find another root of P with a precision of 10^{-6}.

Input

- A single integer $2 \leq n \leq 10^6$.
- A list of $n+1$ integers $-10^6 \leq a_0, ..., a_n \leq 10^6$.
- A single non-zero root α of P such that $\alpha \neq \pm 1$.

Output

- A single root $\beta \neq \alpha$ of P. Your answer should have an absolute error of at most 10^{-6}.

Examples

Input	Output
2 1 3 1 -0.38196601	-2.61803398

Problem 2: Wicked Waves

Difficulty: ★ ☆ ☆ ☆ ☆
Key words: Calculus

Winston is a well-known windsurfer who would like to go windsurfing on a windy Wednesday. However, wanting to avoid falling in the water, Winston is wary of tall waves. To determine if the water is worthy of Winston's whereabouts, he wonders on a whim whether you can calculate the height of the tallest waves. It is widely known that windy waves are well-modeled by the equation

$$h(x) = \sin^n(x)\cos^m(x),$$

which winds the problem down to finding the maximum of this wicked wave equation.

Input

- Two space-separated integers $0 \leq n, m \leq 10$, the exponents of sin and cos respectively.

Output

- The maximum height of the waves. Your answer should have an absolute error of at most 10^{-6}.

Examples

Input	Output
1 1	0.5

Input	Output
7 5	0.016989503

Problem 3: Questionable Quadrupling

Difficulty: ★ ★ ☆ ☆ ☆
Key words: Algebra, Probability Theory

One of your friends Quincy has thought of an interesting way to quadruple the outcome of a die roll. Instead of rolling the die once and quadrupling it, he rolls the die four times and adds the rolls together. Given the frequency table of the quadrupled rolls (i.e. for each possible quadrupling outcome q_i, for how many die values a_1, a_2, a_3, a_4 we have $a_1 + a_2 + a_3 + a_4 = q_i$), can you determine the faces of the die? All faces of the die are different integers.

Input

- The first line contains a single integer $1 \leq n < 10^3$, the number of unique quadrupling outcomes.
- Next follow n lines with two space-separated integers $4 \leq q_i \leq 10^5$, the quadrupling outcome, and $1 \leq k_i < 10^4$, for how many tuples of four original rolls the sum q_i is attained. These lines are sorted in increasing order of q_i and the sum of all k_i does not exceed 10^4. It is guaranteed that a die exists with this quadrupling frequency table.

Output

- The first line should contain a single integer m, the number of faces of the original die.
- Next, output m lines containing a single integer f_i, denoting the faces of the die in increasing order.

Examples

Input	Output
1 4 1	1 1

Input	Output
5 4 1 6 4 8 6 10 4 12 1	2 1 3

Problem 4: よ-Actions

Difficulty: ★ ★ ★ ☆ ☆
Key words: Group actions, Category Theory

A group action of a group G on a set X is a map $G \times X \to X$ such that

- for e the identity of G, $e \cdot x = x$,
- for $g, h \in G$, $g \cdot (h \cdot x) = (gh) \cdot x$.

For example, we have the canonical group action of G acting on itself by left multiplication.
Furthermore, we have the notion of an intertwiner of groups actions. Suppose we have the group G acting on X and on Y, then an intertwiner is a map $f : X \to Y$ such that for $g \in G$ and $x \in X$,

$$f(gx) = gf(x).$$

For a positive integer n, consider the group S_n of permutations of n elements.
Note that S_n is generated by the permutations $(12...n)$ and (12). So, to specify a group action it suffices to specify how $(12...n)$ and (12) act.

Given an action of S_n on the set $X = \{1, ..., m\}$ how many intertwiners are there from the canonical action to the action on X?

Input

- An integer $3 \leq n \leq 10^4$ indicating the permutation group S^n.
- An integer $1 \leq m \leq 10^5$, the number of elements in X.
- A list of m integers $a_1, ..., a_m$ signifying the action of $(12...n)$ on X i.e. $(12...n) \cdot j = a_j$ for $j \in X$.
- A list of m integers $b_1, ..., b_m$ signifying the action of (12) on X i.e. $(12) \cdot j = b_j$ for $j \in X$.

Output

- A single integer k indicating the number of intertwiners.

Examples

Input	Output
3 3 2 3 1 2 1 3	3

Problem 5: Tetration Truncation

Difficulty: ★ ★ ★ ☆ ☆
Key words: Tetration, Number Theory

Tetration is a binary operator based on repeated exponentiation. It is often denoted as

$$a \uparrow\uparrow n := a^{a^{\cdot^{\cdot^{\cdot^{a}}}}},$$

where the tower of exponents contains n copies of a. For example, $2 \uparrow\uparrow 3 = 2^{2^2} = 2^4 = 16$. The problem with explicitly computing tetration is that even for small a and n, the tetration $a \uparrow\uparrow n$ can become very large: the tetration $3 \uparrow\uparrow 4$ already has 3638334640025 digits. In this problem, we are interested in computing the last few digits of otherwise infeasible tetrations. The following theorem may be useful.

Euler's Totient Theorem. *For a natural number n with prime factorization $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ we define Euler's totient function as*

$$\varphi(n) = (p_1 - 1)p_1^{e_1 - 1}(p_2 - 1)p_2^{e_2 - 1} \cdots (p_k - 1)p_k^{e_k - 1}.$$

If n and a are coprime natural numbers, we have

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

Input

- Three space-separated integers $1 \leq a, n, k \leq 10^2$ with $\gcd(a, 10) = 1$. It is given that $a \uparrow\uparrow n$ has at least k digits.

Output

- A string containing the last k digits of $a \uparrow\uparrow n$.

Examples

Input	Output
7 2 3	543

Input	Output
3 4 7	0739387

Problem 6: Complex Caves

Difficulty: ★ ★ ★ ★ ☆
Key words: Graphs, Topology

Tim is an aspiring speleologist[1]. As everyone knows a cave consists of a set of rooms R and a set of tunnels T between rooms. It is also good to keep in mind that caves can be very complicated, for example one might have a tunnel $t \in T$ from a room $r \in R$ to the same room r or multiple tunnels between rooms r and r'. Furthermore, Tim is a very good speleologist, so he can traverse any tunnel in both directions.

Now, Tim would like to rank the caves he has discovered by complexity. Tim generally thinks that caves in which it is easier to get lost are more complex. In fact the most dangerous situation is a loop in your cave, as then you might walk around this loop indefinitely. Thus, Tim introduces the following complexity score: the complexity of a cave is the minimum number of tunnels one needs to fill in order to remove all loops from the cave[2]. Now, Tim wants you to calculate the complexity score of a single connected cave.

Input

- $2 \leq R \leq 1000$ and $R - 1 \leq T \leq 2 \cdot 10^5$ the number of rooms and tunnels in the cave.
- T lines each containing two nodes $1 \leq a, b \leq R$ indicating a tunnel between a and b. Note there might be multiple tunnels between a pair a and b.

Output

- An single integer C, the complexity of this cave.

Examples

[1]Someone who explores caves.

[2]Relating the above story to mathematics: a graph G has a fundamental group $\pi_1(G)$. This group is of the form $\mathbb{Z}^{*n}$. Then, the complexity of a cave corresponds to n.

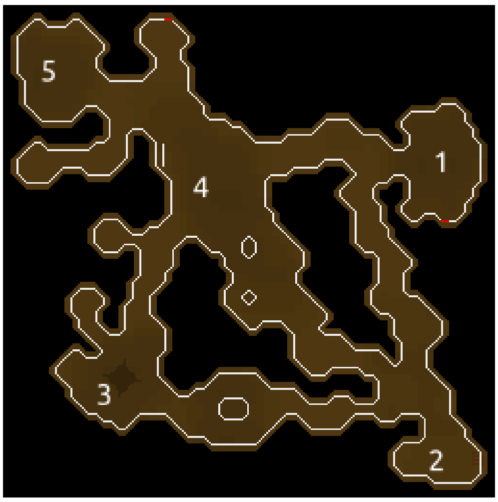

Figure 1: Example given by cave 2

Input	Output
2 3 1 2 1 2 1 2	2

Input	Output
5 6 1 2 1 4 3 2 4 3 4 5 4 2	2

Problem 7: Decimal Determination

Difficulty: ★ ★ ★ ★ ★
Key words: Algebra
Several real numbers have algorithms that allow for efficient computation of a specific digit, without needing to compute the prior digits. For instance, the Bailey–Borwein–Plouffe formula

$$\pi = \sum_{k=0}^{\infty} \left[\frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right]$$

can be exploited to efficiently compute a single digit of π in base 16. In this problem, we instead look at numbers of the form

$$D(A,B,s) := \frac{1}{1 - 0.\underbrace{0\ldots0}_{s \text{ zeroes}}A\underbrace{0\ldots0}_{s \text{ zeroes}}B}$$

with $s \in \mathbb{Z}_{>0}$ and $A, B \in \{0, 1, \ldots, 9\}$. The decimal expansion of such numbers has a nice property that allows for efficient computation of specific digits.

Input

- The first line consists of three space-separated integers A, B, s with $A, B \in \{0, 1, \ldots, 9\}$ and $0 < s \leq 10^4$, the parameters of $D(A, B, s)$ as described above.
- The second line consists of a single integer $0 < d \leq s(s+1)$, the index of the specific digit to calculate.

Output

- A single integer, the d^{th} decimal digit of $D(A, B, s)$ after the decimal point.

Examples

Input	Output
1 1 4 20	5

Input	Output
3 8 500 249876	4